1 Assignment #1

1. Induction Proof(arrangement of lines)

Claim 1. *Let L be a set of lines in general position*[1] *in the plane, with $|L| > 2$. Then, at least one of the regions formed by L is a triangle.*

Proof. Suppose $|L| = 3$. Since all lines are in general position, there exists three points of intersection among the 3 lines, and they form a closed triangle.

For the induction hypothesis, suppose the claim is true for $|L| = n$. Now, assume $|L| = n+1$. Pick any line l from L, and consider the remaining n lines. Since we assumed the claim holds for $|L| = n$, there exists at least one triangle, say ABC. Now, put l back onto L. If l does not intersect the triangle, ABC still forms a triangle, and we are done. If l does intersect the triangle, it crosses precisely 2 sides of ABC. Then, the two intersection points at which l crosses ABC, and one of the vertices of ABC now forms a triangle. □

2. Induction Proof (number theory)

Find the function $f(n)$ that generates, given n, the sum of the square of all positive integers 1 through n.

Expression: $f(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

Proof.

- Base case: $n = 1$. $f(1) = 1/3 + 1/2 + 1/6 = 1$, which is correct.
- Induction hypothesis: Suppose the formula for $f(n)$ is correct for $n \leq k$.
- Consider $n = k + 1$. Then the expression gives:

$$\begin{aligned} f(k+1) &= 1/3(k+1)^3 + 1/2(k+1)^2 + 1/6(k+1) \\ &= 1/3k^3 + 3/2k^2 + 13/6k + 1 \end{aligned}$$

  On the other hand, the definition of sum of squares gives:

$$\begin{aligned} f(k+1) &= f(k) + (k+1)^2 \\ &= 1/3k^3 + 1/2k^2 + 1/6k + (k^2 + 2k + 1) \\ &= 1/3k^3 + 3/2k^2 + 13/6k + 1 \end{aligned}$$

  which completes the proof.

□

[1] Lines are in general position if: (1) no two lines are parallel, and (2) no three lines intersect at one point.

3. Induction Proof (circle map coloring)

Claim 2. *Let C be a set of circles in the plane. Then, the regions formed by C are 2-colorable.*

Proof. Suppose $|C| = 1$. Then, the claim is clearly true.

Suppose the claim is true for $|C| = n$, and consider the case where $|C| = n + 1$. Pick any circle c from C, and put it aside for now. Since there are n remaining circles, we can 2-color the regions using the induction hypothesis.

Now, put c back into C, and consider the coloring. All the regions that does not contain an arc from c still satisfy the coloring property (i.e. no neighboring region shares the same color). All the regions that does contain an arc from c, however, do not satisfy the coloring property. In particular, each of these regions has precisely 1 neighboring region that shares the same color.

The regions that contain an arc from c can be partitioned into two classes: (i) it lies inside c; (ii) it lies outside c. For each region that lies inside c, we flip its color (i.e. change red to black, and vice versa). Now, we claim that the resulting coloring is valid. To see this, let r_1 and r_2 be two arbitrary neighboring regions. If they both lie outside of c, they must be colored in different colors since the border between them comes from $C - \{c\}$. If one lies inside c while the other lies outside, they are now colored in different colors, by the flipping operation. Finally, if they both lies inside c, they were originally colored in different colors before the flipping operation, and after the flipping operation, they now have opposite colors. □

2 Assignment #2

1. Algorithms for Turing Machines

3. Big "Oh" Notation

Definition 1. *If $\exists c, n_0$ such that $f(n) < cg(n)$ for all $n \geq n_0$, then $f(n) \in O(g(n))$.*

Definition 2. *If $\exists c, n_0$ such that $f(n) \geq cg(n)$ for all $n \geq n_0$, then $f(n) \in \Omega(g(n))$.*

(a) $100n + \log n = O(n + \log^2 n)$

We know $100n = O(n)$ and $\log n = O(\log^2 n)$. Lemma 3.1 then gives the sum is also $O(n + \log^2 n)$.

(b) $\log n = \Theta(\log n^2)$

We have $\log n^2 = 2 \log n = \Theta(\log n)$. (Note that Θ works both ways)

(c) $n^2/\log n = \Omega(n \log^2 n)$

Multiplying both sides by $\log n$ gives n^2 and $n \log^3 n$. Assuming the base of the logarithm is 3, $c = 1, n_0 = 27$ shows $n^2 = \Omega(n \log^2 n)$.

(d) $\log^{\log n} n = \Omega(n/\log n)$

Multiplying both sides by $\log n$ gives $\log^{\log n+1} n$ and n. We now show $\log^{\log n+1} n = \Omega(n)$. By plugging in $c = 1$ and $n_0 = 8$, the claim holds.

(e) $n^{0.5} = \Omega(\log^5 n)$

Just plug in $c = 1$ and $n_0 = 100$ into the definition completes the proof.

(f) $n2^n = O(3^n)$

Just plug in $c = 1$ and $n_0 = 10$ into the definition completes the proof.

4. Minimum Spanning Trees

Claim 3. *Let S be a set of $n > 2$ points in the plane in general position, such that $S = B \cup R$ and $B \cap R = \emptyset$ (B for blue, and R for red). Then, for every pair $b \in B$ and $r \in R$ that has the minimum distance between a blue point and a red point, there is a minimum spanning tree of S containing (b, r).*

Proof. Let d denote the minimum distance between a blue point and a red point. Take any pair of points $b \in B$ and $r \in R$ such that $distance(b, r) = d$. We shall show that there is an MST(S) that contains (b, r) as an edge.

First, take an arbitrary MST T. If T contains (b, r), we are done. So we assume otherwise. Now, consider the unique $b - r$ path P in T(because T is a tree, this path is unique). Since $b \in B$ and $r \in R$, this path must contain at least one edge, say (b', r'), whose endpoints are colored differently. We remove (b', r') from T, and then insert (b, r) to obtain the resulting graph T'. It suffices to show that T' is also an MST of S.

To show T' is a spanning tree, observe that the deletion of (b', r') disconnects the original spanning tree T into 2 connected components (2 trees, in particular). Then, by inserting (b, r), the 2 components are now connected. Furthermore, joining 2 trees by an edge cannot introduce a cycle, and hence the resulting graph T' is a tree.

Finally, to show T' is a minimum spanning tree, see how the weight of the tree changed:

$$w(T') = w(T) - distance(b', r') + distance(b, r)$$

Since $distance(b, r) = d$, it must be that $distance(b, r) \leq distance(b', r')$, and therefore $w(T') \leq w(T)$. Since T was an MST, it follows that T' must also be minimum. □

3 Random Problems

One Mouth Theorem

Claim 4. *A* mouth *of a non-convex polygon P is 3 consecutive vertices a, b, and c, such that the closed triangle abc does not contain any vertex of P other than a, b, and c. For every non-convex polygon, it contains at least one mouth.*

Proof. Consider the convex hull $CH(P)$. Since P is non-convex, there is at least one edge (a, b) of $CH(P)$ that doesn't belong to P. Now, consider the sequence of vertices along the boundary of P that aren't contained in $CH(P)$, starting from a to b. This in turn forms a simple polygon Q. Recall the two-ear theorem from previous lecture:

> *"Except for triangles, every simple polygon contains at least two non-overlapping ears."*

Thus, Q contains at least 1 ear, which corresponds to the mouth of P. □